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# The role of initial conditions in the ageing of the long-range spherical model

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**Abstract**

The kinetics of the long-range spherical model evolving from various initial states is studied. In particular, the large-time auto-correlation and auto-response functions are obtained, for classes of long-range-correlated initial states, and for magnetized initial states. The ageing exponents can depend on certain qualitative features of initial states. We explicitly find the conditions for the system to cross over from ageing classes that depend on the initial conditions to those that do not.

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## 1. Introduction

Slowly-relaxing thermodynamic systems have been intensely studied for decades now, and for numerous reasons. One of the main objectives is to understand genuine non-equilibrium states: effecting their description, finding and categorizing the large-scale properties, identifying the necessary qualitative features of each universality class, determining sufficient exponents that would enable us to label the classes, and answering a host of other similar questions usually posed in equilibrium statistical physics.

Systems that exhibit phase transitions relax slowly and may not equilibriate when either quenched to critical temperature or below. For critical quench the relaxation is inhibited due to long-range correlations [1], while when quenched from above to below the critical temperature, it is due to the competition between locally-equilibriated domains for a global equilibrium state [2]. In either of these cases, after waiting for a long time $t_w$, much longer than the transient microscopic timescales, these systems which are not in equilibrium are expected to become statistically scale invariant, when scaled by a characteristic time-dependent lengthscale $L(t)$. The length grows in time algebrically, $L(t) \sim t^{1/z}$, defining the dynamic exponent $z$. The scaling hypothesis, if applicable, proposes that the two-point functions of observables, say correlation- or response- function, take the form

$$\langle \mathcal{O}_i(x,t)\mathcal{O}_j(y,t_w)\rangle = t_w^{-a_{ij}} F_{ij}\left(\frac{|x-y|}{t_w^{1/z}}, \frac{t}{t_w}\right), \tag{1}$$

for large observation time $t > t_w$; where $\mathcal{O}_i(x, t)$ refers to some local observable at time $t$ and position $x$, $a_{ij}$ is a universal exponent, $F_{ij}$ is a scaling function. Here it is assumed that the system is translational- and rotational- invariant. Furthermore, in the ageing regime, namely when $t \gg t_w$, the asymptotic behaviour of the function $F_{ij}$ is as follows,

$$F_{ij}(0, t/t_w) \sim (t/t_w)^{-\lambda_{ij}/z}, \tag{2}$$

and defines an ageing exponent $\lambda_{ij}$. The exponent $a_{ij}$ though may not be an independent exponent. For instance, when $t \approx t_w$, and $|x - y| \ll L(t)$, and further under the conditions that would establish equilibrium correlations of size $L(t)$ in time $t$, the two-point function of observables at critical quench would behave as $\langle \mathcal{O}_i(x, t)\mathcal{O}_j(y, t_w)\rangle \sim |x - y|^{-\Delta_{ij}}$, where $\Delta_{ij}$ is an equilibrium exponent. If this is so for (nearly) equal-time observables, then consistency with scaling hypothesis would imply $a_{ij} = \Delta_{ij}/z$, and hence is not an independent exponent.

The scaling hypothesis holds in many slowly-relaxing systems, and is tested experimentally and theoretically [3]. The long-range spherical model is one of the few models that can be solved exactly. This model when evolved from an uncorrelated initial state shows scaling behaviour at large times [4]. Initial conditions are known to affect the large-scale properties in some short-range models [5–13]. So it would be interesting to know how, in general, the initial conditions influence the scaling behaviour, and in particular, that of long-range models. It is this issue that we address in this paper, after having solved the non-conserved long-range spherical model evolving from different initial states. We evaluate the correlation and the response functions for various initial conditions, and determine their scaling behaviour, and also find how the ageing exponents depend on certain features of the initial state. In some cases the $a_{ij}$ exponents turn out to be independent, implying that the spacial correlations within regions of size $L(t)$ after a time $t$ are not governed by canonical equilibrium fluctuations.

The layout of the paper is as follows. In the next section, the model is defined and an outline of the formal solution is provided. In section 3, the scaling behaviour of auto-correlation and auto-response functions, with long-range initial correlations is investigated. In section 4, the model is analysed with an initial state that has no correlations but has non-vanishing magnetization. Section 5 concludes with a brief summary and a few remarks.

## 2. Long-range spherical model

A spherical model is exactly solvable not only in equilibrium but also when evolving with relaxational dynamics [14]. The techniques used in solving this model can easily be extended to a long-range spherical model [4]. In this section, we first define the long-range model and then solve for the correlation and response functions.

### 2.1. The model

The equilibrium long-range spherical model [15] is described by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{x,x'} J_{x,x'} S_x S_{x'}, \tag{3}$$

where the coupling between the spins is long range and is given by

$$J_{x,0} = J_c|x|^{-(d+\sigma)}, \tag{4}$$

such that $\sum_x J_{x,0} = 1$, which in turn fixes the constant $J_c$. The variable $S_x$ respects the spherical constraint $\widehat{Y}[S] = 0$, where

$$\widehat{Y}[S] = \frac{1}{N}\sum_{x\in\Lambda} S_x(t)^2 - 1, \tag{5}$$

and, $x$ runs over the sites of the $d$-dimensional lattice $\Lambda$, and $N$ is the total number of sites on the lattice.

The coarse-grain dynamics near the critical temperature or the coarsening dynamics when quenched from above to below $T_c$, for the non-conserved order-parameter $S_x$ is given by the following Langevin equation:

$$\partial_t S_x(t) = -\left.\frac{\delta\mathcal{H}}{\delta S_x}\right|_{S_x\to S_x(t)} - S_x(t)\widehat{Z}(t) + \eta_x(t), \tag{6}$$

$$\langle \eta_x(t)\eta_{x'}(t')\rangle = 2T\delta(t-t')\delta_{x,x'}, \tag{7}$$

where $\widehat{Z}$ is the Langrange multiplier that is determined by the constraint. Assigning Stratonovich convention to the stochastic dynamics will lead the constraint to the expression,

$$\widehat{Z}(t) = -\frac{1}{N}\left(2\mathcal{H}(t) - \sum_x S_x(t)\eta_x(t)\right). \tag{8}$$

### 2.2. *Formal solution*

We will first separate the Gaussian and non-Gaussian terms in equation (6) so as to solve. We will see that the fluctuations of the Lagrange multiplier can be neglected when restricted to local observables in the limit $N \to \infty$. Furthermore, the spherical constraint can be replaced by a mean-spherical constraint in this thermodynamic limit. We will then explicitly write the solution in the Gaussian regime using this mean-spherical constraint.

*2.2.1. Fluctuations of the Lagrange multiplier.* The fluctuating part of $\widehat{Z}(t)$ is of $O(1/\sqrt{N})$, and hence $S_x$ can be written as

$$S_x(t) = s_x(t) + \frac{1}{\sqrt{N}}\xi_x(t), \tag{9}$$

where $s_x$ captures only the Gaussian fluctuations while $\xi_x$ accounts for the remaining non-Gaussian fluctuations.

Within the Gaussian approximation the mean-square fluctuations in $\widehat{Y}[s]$ can be estimated from the following relation, obtained by using Wick's theorem and translation invariance:

$$\langle \widehat{Y}^2[s]\rangle - \langle \widehat{Y}[s]\rangle^2 \approx \frac{2}{N}\sum_x C_{x,0}^2(t,t), \tag{10}$$

where $C_{x,0}(t,t)$ is the equal-time spin–spin correlation function. In the case of uncorrelated initial state, the late-time behaviour of the correlation function is as given below [4],

$$C_{x,0}(t,t) \sim \begin{cases} G(|x|t^{-1/\sigma}), & T < T_c \\ t^{1-d/\sigma}G(|x|t^{-1/\sigma}), & T = T_c \end{cases}, \tag{11}$$

where, $G(x) \propto \int_k \exp(ik\cdot x - 2|k|^\sigma)$. Hence we obtain,

$$\langle \widehat{Y}^2[s]\rangle - \langle \widehat{Y}[s]\rangle^2 \propto \begin{cases} t^{d/\sigma}/N, & T < T_c \\ t^{2-d/\sigma}/N, & T = T_c \end{cases}. \tag{12}$$

Therefore, in the limit $N \to \infty$ these fluctuations become negligible for late times $t < t^* \sim N^{\sigma/d}$ [16], both at and below the critical temperature $T_c$. Even at $T_c$ the threshold time, $t^*$, is unchanged since the phase transition occurs only for $0 < \sigma < d$. Thus, taking first the $N \to \infty$ limit is a necessary condition to have the same asymptotic behaviour for both spherical and mean-spherical models in the Gaussian regime.

Substituting the decomposition of spin variables, as given in equation (9), into equation (5) and keeping only the leading subdominant terms yields the expression

$$\widehat{Y}[S] \approx \langle\widehat{Y}[s]\rangle + \frac{1}{N}\left[\sum_x \left(s_x^2 - \langle s_x^2\rangle\right) + \frac{2}{\sqrt{N}}\sum_x s_x \xi_x\right]. \tag{13}$$

Here the non-Gaussian term is of the same order in $N$ as the Gaussian fluctuations, and hence can play a role in certain global observables.

*2.2.2. Solution in the Gaussian regime.* The Gaussian approximation of the Langevin equation (6) in the Fourier space is given as

$$\partial_t s_k(t) = -(\omega_k + Z(t))s_k(t) + \eta_k(t), \tag{14}$$

where $\omega_k$ is the fourier transform of $-J_{x,0}$, and the noise $\eta_k(t)$ has the variance,

$$\langle \eta_k(t)\eta_{k'}(t')\rangle = 2T\delta(t-t')N\delta_{k+k'}. \tag{15}$$

In the continuum limit, $N\delta_k$ gets replaced by $(2\pi)^d\delta(k)$, and $N^{-1}\sum_k$ is replaced by $(2\pi)^{-d}\int \mathrm{d}^d k$. In this limit, for small $k$, the function $\omega_k \to \widetilde{c}_0 + \widetilde{c}_1|k|^\sigma + \widetilde{c}_2|k|^2 + O(|k|^4)$, where $\widetilde{c}_n$ are lattice-dependent constants. We shall restrict $\sigma < 2$, for $\sigma \geqslant 2$ the relevant model is the usual short-range spherical model. Further, without any loss of generality we shall set $\widetilde{c}_0 = 0$ by absorbing it into $Z(t)$.

The solution of the above equation is

$$s_k(t) = \frac{\mathrm{e}^{-\omega_k t}}{\sqrt{g(t)}}\left[s_k(0) + \int_0^t \mathrm{d}\tau\, \mathrm{e}^{\omega_k \tau}\sqrt{g(\tau)}\eta_k(\tau)\right], \tag{16}$$

where the yet to be determined constraint function $g(t) = \exp\left(2\int_0^t \mathrm{d}\tau Z(\tau)\right)$. It is often convenient to write the above equation as

$$s_k(t) = R_k(t,0)s_k(0) + \int_0^t \mathrm{d}\tau\, R_k(t,\tau)\eta_k(\tau), \tag{17}$$

where Green's function $R_k(t,t')$ satisfies the equation

$$(\partial_t + \omega_k + Z(t))R_k(t,t') = \delta(t-t'), \tag{18}$$

with the initial condition $R_k(t,t') = 0$ for $t < t'$. It is easy to see that $R_k(t,t')$ is also the Fourier transform of the response function $R_{x,0}(t,t')$, and is explicitly given as

$$R_k(t,t') = \mathrm{e}^{-\omega_k(t-t')}\sqrt{\frac{g(t')}{g(t)}}\theta(t-t'), \tag{19}$$

where $\theta(t)$ is the step function.

If the initial conditions for the spins $s_x$ are chosen to be translation invariant with a variance,

$$\langle s_k(0)s_{k'}(0)\rangle = N\delta_{k+k'}C_k(0,0). \tag{20}$$

then, using equation (16), we obtain the correlation function

$$\langle s_k(t)s_{k'}(t')\rangle = N\delta_{k+k'}C_k(t,t'), \tag{21}$$

where

$$C_k(t,t') = \frac{g(t_m)}{\sqrt{g(t)g(t')}} \mathrm{e}^{-\omega_k|t-t'|} C_k(t_m,t_m), \tag{22}$$

and $t_m = \min(t,t')$, and the equal-time correlation function

$$C_k(t,t) = \frac{\mathrm{e}^{-2\omega_k t}}{g(t)} \left[ C_k(0,0) + 2T \int_0^t \mathrm{d}\tau\, \mathrm{e}^{2\omega_k \tau} g(\tau) \right]. \tag{23}$$

Note that $C_k(t,t)$ is also the Fourier transform of the correlation function $C_x(t,t) = \langle S_{x+y}(t) S_x(t) \rangle$. We now impose the mean-spherical constraint, $\langle \widehat{Y}[s] \rangle = 0$, which then implies that the following condition should hold at all times:

$$\frac{1}{N} \sum_k C_k(t,t) = 1. \tag{24}$$

This condition not only fixes $g(t)$ but also restricts the choice of initial conditions. Substituting expression (23) into the above equation gives $g(t)$ to be the solution of the following Volterra equation:

$$g(t) = A(t) + 2T \int_0^t \mathrm{d}\tau f(t-\tau) g(\tau), \tag{25}$$

with $g(0) = 1$, where

$$A(t) = \frac{1}{N} \sum_k \mathrm{e}^{-2\omega_k t} C_k(0,0), \tag{26}$$

and

$$f(t) = \frac{1}{N} \sum_k \mathrm{e}^{-2\omega_k t}. \tag{27}$$

Note that $A(0) = 1$ due to the spherical constraint. The function $A(t)$ depends on the initial correlations, and hence $g(t)$. We shall solve equation (25) and determine $g(t)$ for various initial conditions in latter sections, and explicitly evaluate the correlation and response functions.

The non-Gaussian variable $\xi_x(t)$ can be completely expressed in terms of Gaussian observables as has been done recently for the short-range spherical model [10]. But we shall not express this here, since we are interested in the ageing properties of local observables, for which the non-Gaussian corrections vanish in the limit $N \to \infty$.

## 3. Long-range-correlated initial conditions

In this section, we will evaluate the auto-correlation and auto-response functions, when evolved from a long-range-correlated initial state. To this end, we will first calculate the constraint function.

### *3.1. Constraint function*

The large-time solution of equation (25) for the constraint function $g(t)$ not only depends on the temperature $T$ but also on the form of the initial correlation function $C_k(0,0)$. The case with uncorrelated initial conditions in this model was studied recently [4]. We now investigate the behaviour when evolved from a class of initial states with long-range spin–spin correlations. In other words, we choose $C_k(0,0)$ such that its small $k$ behaviour is given by

$$C_k(0,0) \approx c_0 |k|^{\sigma_0}, \tag{28}$$

**Table 1.** Values of $g_0$ and $\psi$, as defined in equation (32), characterizing the asymptotics of $g(t)$ in various critical regimes.

| Regime | Conditions | $g_0(\sigma, \sigma_0, d)$ | $\psi(\sigma, \sigma_0, d)$ |
|---|---|---|---|
| I | $d/2 < \sigma < d$, $-d < \sigma_0 < \sigma - d$ | $\frac{A_1 B_0}{A_0 \Gamma(-\sigma_0/\sigma)}$ | $-1 - \sigma_0/\sigma$ |
| II | $0 < \sigma < d/2$, $-d < \sigma_0 < \sigma - d$ | $\frac{A_1 B_0}{A_2 \Gamma(2-(d+\sigma_0)/\sigma)}$ | $1 - (d + \sigma_0)/\sigma$ |
| III | $d/2 < \sigma < d$, $\sigma_0 > \sigma - d$ | $\frac{A_1 B_1}{A_0 \Gamma(-1+d/\sigma)}$ | $-2 + d/\sigma$ |
| IV | $0 < \sigma < d/2$, $\sigma_0 > \sigma - d, \sigma_0 > -\sigma$ | $\frac{A_1 B_1}{A_2}$ | 0 |
| V | $0 < \sigma < d/2$, $\sigma_0 > \sigma - d, \sigma_0 < -\sigma$ | $\frac{A_1 B_1}{A_2}$ | 0 |

and also satisfies equation (24). In the real space, the large-distance initial correlations become $C_x(0, 0) \sim c_0'|x|^{-(d+\sigma_0)}$. The exponent $\sigma_0$ is restricted by the condition $d + \sigma_0 > 0$, on the grounds that the correlations should decrease with increase in distance. A short-range spherical model with a similar initial state was also studied recently [8].

The Laplace transform of equation (25) results in the expression

$$g_L(p) = \frac{A_L(p)}{1 - 2T f_L(p)}, \tag{29}$$

where, $g_L(p), A_L(p)$ and $f_L(p)$ are the Laplace transforms of $g(t), A(t)$ and $f(t)$, respectively. The small $p$ behaviour of $g_L(p)$ is sufficient for deducing the large-$t$ properties of $g(t)$. The function $f_L(p)$, for small $p$, is given as

$$f_L(p) = -A_0 p^{-1+d/\sigma} + \sum_{n=0}^{\infty} A_{n+1}(-p)^n \tag{30}$$

where $A_0 = |\Gamma(1 - d/\sigma)|\widetilde{A}_0$ and $A_n = N^{-1}\sum_k (2\omega_k)^{-n} - \int_k (2\widetilde{c}_1|k|^\sigma)^{-n}$, and $\widetilde{A}_n = \int_k \exp(-2\widetilde{c}_1|k|^\sigma)|c_0 k|^{n\sigma_0}$. Similarly, $A_L(p)$ is given by

$$A_L(p) = -B_0 p^{-1+(d+\sigma_0)/\sigma} + \sum_{n=0}^{\infty} B_{n+1}(-p)^n \tag{31}$$

where $B_0 = |\Gamma(1 - (d + \sigma_0)/\sigma)|c_0\widetilde{A}_1$ and $B_n = N^{-1}\sum_k (2\omega_k)^{-n} C_k(0, 0) - \int_k (2\widetilde{c}_1|k|^\sigma)^{-n} c_0|k|^{\sigma_0}$.

There is a phase transition at temperature $T_c = 1/2A_1$. At the critical temperature $T_c$, the small $p$ behaviour of $g_L(p)$ can easily be obtained by substituting equations (30) and (31) into (29). In this limit, $g_L(p) \approx g_0\Gamma(\psi + 1)p^{-(\psi+1)}$ and hence the large-time form of $g(t)$ is given by

$$g(t) \approx g_0(\sigma, \sigma_0, d)t^{\psi(\sigma,\sigma_0,d)}, \tag{32}$$

where the constant $g_0(\sigma, \sigma_0, d)$ and the exponent $\psi(\sigma, \sigma_0, d)$ depend on the values of $\sigma$ and $\sigma_0$, for a given $d$. There are four regimes, as specified in table 1, each distinguished by the form of $\psi$. Though the form of $g(t)$ in case IV and case V is the same, the two cases are listed separately for, as we shall see later, they get distinguished by ageing exponents and the fluctuation–dissipation ratio.

For $T < T_c$, in the small $p$ region the function $g_L(p) \approx (1 - T/T_c)^{-1} A_L(p)$, and hence

$$g(t) \approx \widetilde{A}_1 \left(1 - \frac{T}{T_c}\right)^{-1} t^{-(d+\sigma_0)/\sigma}. \tag{33}$$

### 3.2. Auto-correlation and auto-response functions

We first write the auto-correlation and auto-response functions in terms of $g(t)$, $A(t)$ and $f(t)$, and then analyse their late-time behaviour. The correlation function is given by

$$C_x(t, t_w) := \langle S_{x+y}(t) S_y(t_w)\rangle = \frac{1}{N}\sum_k \mathrm{e}^{ik\cdot x} C_k(t, t_w), \tag{34}$$

and hence the expression for the auto-correlation function, $C(t, t_w) := C_{x=0}(t, t_w)$, using equations (22) and (23), becomes

$$C(t, t_w) = \frac{1}{\sqrt{g(t)g(t_w)}}\left[A\left(\frac{t+t_w}{2}\right) + 2T\int_0^{t_w} \mathrm{d}\tau f\left(\frac{t+t_w}{2} - \tau\right) g(\tau)\right]. \tag{35}$$

The spin response to magnetic field is obtained by perturbing the Hamiltonian, $\mathcal{H}(t) \to \mathcal{H}(t) - \int^t \mathrm{d}\tau \sum_x S_x(\tau) h_x(\tau)$, with an impulse of magnetic field at time $t_w$, and is given by

$$\begin{aligned} R_x(t, t_w) &:= \left.\frac{\delta\langle S_{x+y}(t)\rangle_h}{\delta h_y(t_w)}\right|_{h=0} \\ &= \frac{1}{N}\sum_k \mathrm{e}^{ik\cdot x}\,\mathrm{e}^{-\omega_k(t-t_w)}\sqrt{\frac{g(t_w)}{g(t)}}, \end{aligned} \tag{36}$$

for $t \geqslant t_w$, and zero otherwise. The auto-response function, $R(t, t_w) := R_{x=0}(t, t_w)$, therefore takes the form

$$R(t, t_w) = \sqrt{\frac{g(t_w)}{g(t)}}\, f\left(\frac{t - t_w}{2}\right). \tag{37}$$

#### 3.2.1. *Critical quench.*

The asymptotic behaviour of $f(t) \sim \widetilde{A}_0 t^{-d/\sigma}$ and $A(t) \sim \widetilde{A}_1 t^{-(d+\sigma_0)/\sigma}$, is independent of temperature. Making use of these asymptotics and that of $g(t)$ at $T_c$ as given in equation (32), expression (35) at large $t$ and $t_w$ reduces to

$$C(t, t_w) = 2^{d/\sigma}\widetilde{A}_0 t_w^{1-d/\sigma}\left[t_w^{-(1+\psi+\sigma_0/\sigma)} F_{1C}(x) + F_{2C}(x)\right], \tag{38}$$

where $x = t/t_w$ and

$$F_{1C}(x) = \frac{\widetilde{A}_1}{g_0 \widetilde{A}_0} 2^{\sigma_0/\sigma} x^{-\psi/2} (x+1)^{-(d+\sigma_0)/\sigma}, \tag{39}$$

$$F_{2C}(x) = 2T_c x^{-\psi/2}\int_0^1 \mathrm{d}v (x + 1 - 2v)^{-d/\sigma} v^{\psi}. \tag{40}$$

Thus the auto-correlation function has the scaling form

$$C(t, t_w) = 2^{d/\sigma}\widetilde{A}_0 t_w^{-b} f_C(x), \tag{41}$$

with the following three possibilities:

(i) $b = -1 + d/\sigma$ and $f_C(x) = F_{1C}(x) + F_{2C}(x)$, when $1 + \psi + \sigma_0/\sigma = 0$;
(ii) $b = -1 + d/\sigma$ and $f_C(x) = F_{2C}(x)$, when $1 + \psi + \sigma_0/\sigma > 0$;
(iii) $b = \psi + (d + \sigma_0)/\sigma$ and $f_C = F_{1C}(x)$, when $1 + \psi + \sigma_0/\sigma < 0$.

**Table 2.** Ageing exponents and asymptotic FDR in various critical regimes.

| Regime | $b$ | $\lambda_C$ | $a$ | $\lambda_R$ | $X^{\infty}$ |
|---|---|---|---|---|---|
| I | $-1+d/\sigma$ | $d+(\sigma_0-\sigma)/2$[a] | $-1+d/\sigma$ | $-(\sigma+\sigma_0)/2$ | $\delta_{\sigma+\sigma_0,0}$ |
| II | 1 | $(d+\sigma+\sigma_0)/2$ | $-1+d/\sigma$ | $(d+\sigma-\sigma_0)/2$ | 0 |
| III | $-1+d/\sigma$ | $3d/2-\sigma$ | $-1+d/\sigma$ | $3d/2-\sigma$ | $1-\sigma/d$ |
| IV | $-1+d/\sigma$ | $d$ | $-1+d/\sigma$ | $d$ | $1/2$ |
| V | $(d+\sigma_0)/\sigma$ | $d+\sigma_0$ | $-1+d/\sigma$ | $d$ | 0 |

[a] Except for the fine-tuned case, where $\lambda_C = d$.

One of these possibilities occurs in each of the five regimes mentioned in table 1. In the limit $x \to \infty$

$$f_C(x) \sim x^{-\lambda_C/z}, \tag{42}$$

where the dynamic exponent $z = \sigma$. The values of $b$ and $\lambda_C$ in various regimes are listed explicitly in table 2, with one fine-tuned exception in regime I. This exceptional case is when not only $\sigma_0 = -\sigma$ but also the amplitude of the initial correlation is tuned such that it exactly corresponds to correlations of the equilibrium long-range spherical model. In this fine-tuned case, the system is always at equilibrium and the value of the ageing exponent $\lambda_C = d$.

Similarly, using the asymptotics of $f(t)$ and $g(t)$ in equation (37) we obtain the expression for the response function for large times as

$$R(t, t_w) = 2^{d/\sigma}\widetilde{A}_0 t_w^{-d/\sigma} x^{-\psi/2}(x-1)^{-d/\sigma}, \tag{43}$$

where $x \neq 1$. Here again the auto-response function takes the expected scaling form

$$R(t, t_w) = 2^{d/\sigma}\widetilde{A}_0 t_w^{-1-a} f_R(x), \tag{44}$$

where $a = -1 + d/\sigma$ and $f_R(x) = x^{-\psi/2}(x-1)^{-d/\sigma}$, which in the limit $x \to \infty$ behaves as

$$f_R(x) \sim x^{-\lambda_R/z}, \tag{45}$$

with $\lambda_R = d + \sigma\psi/2$. The values of $b$ and $\lambda_R$ in all the five regimes are listed explicitly in table 2.

Another interesting quantity to look at is the fluctuation–dissipation ratio (FDR). A deviation of this ratio from unity indicates that the state is nonequilibrium. FDR, which is defined as

$$X(t, t_w) := T\frac{R(t, t_w)}{\partial_{t_w} C(t, t_w)}, \tag{46}$$

can be obtained explicitly up on using equations (38) and (43), and is given by the expression

$$X(t, t_w) = T_c \frac{x^{-\psi/2}(x-1)^{-d/\sigma}}{t_w^{-(1+\psi+\sigma_0/\sigma)}\widetilde{F}_{1C}(x) + \widetilde{F}_{2C}(x)}, \tag{47}$$

where $x \neq 1$, and

$$\widetilde{F}_{1C}(x) = -\left(\psi + \frac{d+\sigma_0}{\sigma}\right)F_{1C}(x) - x\frac{\mathrm{d}}{\mathrm{d}x}F_{1C}(x), \tag{48}$$

$$\widetilde{F}_{2C}(x) = \left(1 - \frac{d}{\sigma}\right)F_{2C}(x) - x\frac{\mathrm{d}}{\mathrm{d}x}F_{2C}(x). \tag{49}$$

The large $x$ behaviour of $\widetilde{F}_{1C}(x)$, using equations (39) and (48), is easily obtainable and is given by

$$\widetilde{F}_{1C}(x) \sim \frac{\widetilde{A}_1}{g_0\widetilde{A}_0} 2^{\sigma_0/\sigma}(-k_0)x^{-(k_1+(d+\sigma_0)/\sigma)}, \tag{50}$$

where either $k_0 = k_1 = \psi/2$ when $\psi \neq 0$, or $k_0 = (d+\sigma_0)/\sigma$ and $k_1 = 1$ when $\psi = 0$. Similarly, the large-$x$ behaviour of $\widetilde{F}_{2C}(x)$, using equations (40) and (49), for $\psi \neq 0$ is given by

$$\widetilde{F}_{2C}(x) \sim T_c \frac{2+\psi}{1+\psi} x^{-(\psi/2+d/\sigma)}, \tag{51}$$

while for $\psi = 0$,

$$\widetilde{F}_{2C}(x) = T_c[(x+1)^{-d/\sigma} + (x-1)^{-d/\sigma}]. \tag{52}$$

Using the above expressions for $\widetilde{F}_{1C}(x)$ and $\widetilde{F}_{2C}(x)$ in equation (47), it is easy to evaluate the asymptotic FDR

$$X^{\infty} := \lim_{t_w\to\infty}\big(\lim_{t\to\infty} X(t,t_w)\big) = \lim_{x\to\infty}\big(\lim_{t_w\to\infty} X(t,t_w)|_{x=t/s}\big). \tag{53}$$

The asymptotic FDR in all cases is obtained as specified in table 2.

A few remarks on the ageing exponents follow. Both regimes III and IV are independent of initial conditions, and the exponents are the same as in the case of the uncorrelated initial state [4]. Note that in both these regimes the condition $\sigma + \sigma_0 > 0$ is satisfied. A physical interpretation of this inequality is that the initial correlations are weaker than the equilibrium correlations towards which the system is evolving. This is easily deduced by comparing equilibrium correlations $C_{\rm eq}(x) \sim |x|^{-(d-\sigma)}$ [15] with initial correlations $C_{\rm in}(x) \sim |x|^{-(d+\sigma_0)}$. Also note that the auto-correlation exponent $b$ in the case of the uncorrelated initial state [4] is not independent, since we can obtain $C_{\rm eq}(x) \sim |x|^{-bz}$, when $b = -1 + d/\sigma$ is chosen.

In regimes II and V, $\sigma + \sigma_0 < 0$, and hence it can be expected that the ageing exponents will be influenced by the initial conditions, which indeed is the case.

Regime I is where $\sigma + \sigma_0$ can be positive, or negative, or even zero. Here some of the exponents, namely $a$ and $b$, are blind to initial conditions while others, namely $\lambda_C$ and $\lambda_R$ are not. The earlier-mentioned physical picture seems to fail in this regime. When $\sigma+\sigma_0 = 0$, the asymptotic FDR $X^{\infty} = 1$, though in general $\lambda_C \neq \lambda_R$. The only exception is the fine-tuned case where the system is at equilibrium to begin with. In all the regimes which depend on initial conditions, $\lambda_C < \lambda_R$, suggesting that the correlations decay slower than the responses.

*3.2.2. Phase ordering.* We shall briefly discuss the scaling behaviour when the system evolves from the long-range initial conditions below critical temperature. Substituting equation (33) into expression (35) results in the following late-time auto-correlation function when $T < T_c$:

$$C(t,t_w) = \left(1 - \frac{T}{T_c}\right)\left(\frac{(x+1)^2}{4x}\right)^{-(d+\sigma_0)/2\sigma}. \tag{54}$$

Thus scaling behaviour is exhibited even while coarsening; with the exponents $b = 0$ and $\lambda_C = (d+\sigma_0)/2$.

Similarly, substituting equation (33) into expression (37) gives the late-time auto-response function as

$$R(t,t_w) = 2^{d/\sigma}\widetilde{A}_0 t_w^{-d/\sigma} x^{(d+\sigma_0)/2\sigma}(x-1)^{-d/\sigma}, \tag{55}$$

and which shows scaling behaviour with the exponents $a = -1 + d/\sigma$ and $\lambda_R = (d-\sigma_0)/2$. It is easy to see, from equations (54) and (55), that in the phase-ordering regime the asymptotic FDR vanishes, or rather $X(t,t_w) \to 0$ as $t_w \to \infty$ for any fixed $x$.

## 4. Magnetized initial conditions

In this section, we evaluate the late-time auto-correlation and -response functions, within the Gaussian approximation, when the system evolves at critical temperature from an initial state having a non-zero magnetization,

$$\frac{1}{N}\sum_x \langle s_x(0)\rangle = \frac{1}{N}\langle s_k(0)\rangle|_{k=0} = m_0, \tag{56}$$

and uncorrelated spin–spin correlations,

$$\langle s_x(0)s_y(0)\rangle = \left(1 - m_0^2\right)\delta_{x,y} + m_0^2, \tag{57}$$

which respect the spherical constraint.

Recent studies in the short-range spherical model [10, 13] have shown that the non-Gaussian corrections are not relevant for local observables, and only become significant for global observables, when magnetized initial conditions are considered. These results will also hold for the long-range spherical model with magnetized initial conditions.

### 4.1. Constraint function

Equation (57) implies $C_k(0,0) = \left(1 - m_0^2\right) + m_0^2 N\delta_{k,0}$, which upon substituting into equation (26) gives

$$A(t) = \left(1 - m_0^2\right) f(t) + m_0^2. \tag{58}$$

Using the Laplace transform of the above function in equation (29), and then performing an inverse-Laplace transform in the small $p$ region, gives the following late-time behaviour of the constraint function,

$$g(t) \approx g_M(\sigma, d)t^{-2\theta}\left(1 + \frac{t}{t_M}\right), \tag{59}$$

where $t_M = (1-2\theta)\left(1-m_0^2\right)/2T_c m_0^2$, is a time scale associated with the initial magnetization, and

$$\theta = \begin{cases} 0, & 0 < \sigma < d/2 \\ 1 - d/2\sigma, & d/2 < \sigma < d \end{cases}, \tag{60}$$

$$g_M(\sigma, d) = \frac{1 - m_0^2}{4T_c^2} \times \begin{cases} 1/A_2, & 0 < \sigma < d/2 \\ 1/A_0\Gamma(-1 + d/\sigma), & d/2 < \sigma < d \end{cases}. \tag{61}$$

### 4.2. Magnetization

We first evaluate the large-time behaviour of the magnetization, $m(t) := \langle s_{k=0}(t)\rangle/N$, which indeed obeys the expected scaling form [5]. Using equations (16), (56) and (59), we obtain the magnetization as

$$m(t) = \frac{m_0}{\sqrt{g(t)}} \approx \frac{m_0}{\sqrt{g_M}}t^{\theta}\left(1 + \frac{t}{t_M}\right)^{-1/2}. \tag{62}$$

For $t \ll t_M$, the function $m(t) \sim t^{\theta}$, and hence the initial-slip exponent is equal to $\theta$. While for $t \gg t_M$ the asymptotic behaviour of the magnetization is $m(t) \sim t^{\theta-1/2}$. Now the large-time behaviour of the order parameter near the critical temperature is expected to relax as $t^{-\beta/\nu z}$, where $\beta$ and $\nu$ are the equilibrium exponents, and the dynamic exponent $z = \sigma$. For the long-range spherical model [15], $\beta = 1/2$ and

$$\nu = \begin{cases} 1/\sigma, & 0 < \sigma < d/2 \\ 1/(d-\sigma), & d/2 < \sigma < d \end{cases} . \tag{63}$$

Thus in this model the slip exponent is not independent and is given by the relation $\theta = -\beta/\nu z + 1/2$. If we also introduce long-range correlations in the magnetized initial states, then $\theta$ can depend on the classes of initial correlations, and hence can be an independent exponent.

### 4.3. Auto-correlation and auto-response functions

Since the expectation value of the spin is non-zero in the presence of initial magnetization, the Fourier transform of the connected correlation is given by

$$\widetilde{C}_k(t, t_w) = C_k(t, t_w) - N\delta_{k,0} m(t) m(t_w). \tag{64}$$

Therefore, the auto-correlation function, $\widetilde{C}(t, t_w) := \widetilde{C}_{x=0}(t, t_w) = \sum_k \widetilde{C}_k(t, t_w)/N$, is given by

$$\widetilde{C}(t, t_w) = \frac{1}{\sqrt{g(t)g(t_w)}} \left[ \left(1 - m_0^2\right) f\left(\frac{t+t_w}{2}\right) + 2T_c \int_0^{t_w} \mathrm{d}\tau f\left(\frac{t+t_w}{2} - \tau\right) g(\tau) \right]. \tag{65}$$

Using the asymptotic form of $g(t)$, as given in equation (59), and $f(t) \sim \widetilde{A}_0 t^{-d/\sigma}$, and the fact that the above defined $\theta < 1/2$, equation (65) reduces to

$$C(t, t_w) \approx \widetilde{A}_0 2^{d/\sigma} t_w^{1-d/\sigma} F_C(x, u), \tag{66}$$

where

$$F_C(x, u) = \frac{2T_c x^\theta}{\sqrt{(1+ux)(1+u)}} \int_0^1 \mathrm{d}v (x + 1 - 2v)^{-d/\sigma} v^{-2\theta} (1 + uv), \tag{67}$$

$x = t/t_w > 1$ and $u = t_w/t_M$. Hence the value of the exponent $b = -1 + d/\sigma$. In the ageing regime, $x \gg 1$, the above function takes the asymptotic form

$$F_C(x, u) \sim \frac{2T_c x^{\theta - d/\sigma}}{\sqrt{(1+ux)(1+u)}} \left( \frac{1}{1-2\theta} + u \frac{1}{2-2\theta} \right), \tag{68}$$

and has the ageing exponent $\lambda_C$, as listed in the table 3, which depends on whether $t_w \ll t_M$ or $t_w \gg t_M$.

The auto-response function for the case with non-zero initial magnetization, from equations (37) and (59), is obtained as

$$R(t, t_w) = 2^{d/\sigma} \widetilde{A}_0 t_w^{-d/\sigma} F_R(x, u), \tag{69}$$

where

$$F_R(x, u) = \sqrt{\frac{1+u}{1+ux}} (x-1)^{-d/\sigma} x^\theta. \tag{70}$$

Hence the value of the exponent $a = -1 + d/\sigma$. The ageing exponent $\lambda_R$, which again depends on whether $t_w \ll t_M$ or $t_w \gg t_M$, is given in table 3. Both the exponents, $a$ and $b$, do not notice the time scale $t_M$ and are unaffected by the initial magnetization.

The expression for FDR, obtained from equations (66) and (69), reduces to

$$X(t, t_w) = \frac{T_c F_R(x, u)}{(1 - d/\sigma + u\partial_u - x\partial_x) F_C(x, u)}, \tag{71}$$

**Table 3.** Ageing exponents and asymptotic FDR in the Gaussian regime in the presence of initial magnetization.

| Conditions | $a = b$ | $\lambda_C = \lambda_R$ | $X^\infty$ |
|---|---|---|---|
| $t_w \ll t_M, 0 < \sigma < d/2$ | $-1 + d/\sigma$ | $d$ | $1/2$ |
| $t_w \ll t_M, d/2 < \sigma < d$ | $-1 + d/\sigma$ | $3d/2 - \sigma$ | $1 - d/\sigma$ |
| $t_w \gg t_M, 0 < \sigma < d/2$ | $-1 + d/\sigma$ | $d + \sigma/2$ | $2/3$ |
| $t_w \gg t_M, d/2 < \sigma < d$ | $-1 + d/\sigma$ | $(3d - \sigma)/2$ | $d/(d + \sigma)$ |

and results in the following asymptotic value of FDR,

$$X^\infty = \begin{cases} (1-2\theta)/(2-2\theta), & t_w \ll t_M \\ 2(1-\theta)/(3-2\theta), & t_w \gg t_M \end{cases}, \tag{72}$$

which is explicitly given in the table 3.

In the case where $t_w \ll t_M$, the initial magnetization is irrelevant and the ageing exponents are the same as those in the case of the uncorrelated initial state with no initial magnetization. As the waiting time $t_w$ increases and becomes larger than $t_M$, the system will crossover to a new magnetized ageing class.

## 5. Conclusion

To summarize, the correlation and response functions for the non-conserved long-range spherical model are determined for two different sets of initial conditions:

(i) long-range initial correlations with an arbitrary power $\sigma_0 > -d$ and vanishing magnetization;
(ii) non-zero initial magnetization $m_0$, but no initial correlations beyond spherical fluctuations.

We evaluated the ageing exponents corresponding to various ageing classes. The ageing exponents can depend on the class of the initial state, though not on the details of it.

When the long-range initial conditions are considered, the correlations and responses do show scaling behaviour for all $\sigma_0 > -d$, in both the mean-field and the non-trivial cases of the long-range spherical model.

In the mean-field case ($0 < \sigma < d/2$), as the value of $\sigma_0$ increases we distinguish three regimes:

(i) regime II, where $-d < \sigma_0 < \sigma - d$,
(ii) regime V, where $\sigma_0 > \sigma - d$ and $\sigma_0 < -\sigma$,
(iii) regime IV, where $\sigma_0 > \sigma - d$ and $\sigma_0 > -\sigma$.

In both regimes II and V, the initial correlations are stronger than the equilibrium correlations, while in regime IV it is the other way around. We find that the exponents in regimes II and V can depend on $\sigma_0$, whereas this is not the case in regime IV. In other words, we find that as we reduce the initial correlations there is a crossover from the initial-condition-dependent ageing classes to that with no dependence.

In the non-trivial case ($d/2 < \sigma < d$), as the value of $\sigma_0$ increases we distinguish two regimes:

(i) Regime I, where $-d < \sigma_0 < \sigma - d$,
(ii) Regime III, where $\sigma_0 > \sigma - d$.

We find that the exponents in regime I can depend on $\sigma_0$, whereas this is not the case in regime III. Here too there is a crossover from the initial-condition-dependent ageing class to that with no dependence, as $\sigma_0$ goes from below to above $\sigma - d$. In regime III, the initial correlations are weaker than the equilibrium correlations. While in regime I, depending on the value of $\sigma$, it can be weaker, similar, or stronger, though the exponents depend on $\sigma_0$. We lack a simple physical explanation of this dependence on the initial conditions in regime I.

In the case of non-vanishing initial magnetization too, the correlations and responses show scaling behaviour. The initial magnetization $m_0$ introduces a time scale $t_M$, and when the waiting time $t_w \ll t_M$, the ageing classes are same as those obtained with $m_0 = 0$. As $t_w$ increases beyond $t_M$, there is a crossover to the magnetized ageing classes.

Though we have evaluated only auto-correlation and auto-response functions, a similar analysis will suggest that the form of the two-point functions, in the presence of initial correlations, gets modified from that given in equation (1) to the following form,

$$\langle \mathcal{O}_i(x,t)\mathcal{O}_j(y,t_w)\rangle = t_w^{-a_{ij}} F_{ij}\left(\frac{|x-y|}{t_w^{1/z}}, \frac{t}{t_w}\right) + t_w^{-\widetilde{a}_{ij}} \widetilde{F}_{ij}\left(\frac{|x-y|}{t_w^{1/z}}, \frac{t}{t_w}\right), \tag{73}$$

where the second term on the right carries the memory of initial conditions and defines a new exponent $\widetilde{a}_{ij}$ and a scaling function $\widetilde{F}_{ij}$. The function asymptotically behaves as $\widetilde{F}_{ij}(0,x) \sim x^{-\widetilde{\lambda}_{ij}/z}$, and defines another exponent $\widetilde{\lambda}_{ij}$. When $\widetilde{a}_{ij} \leqslant a_{ij}$, then the fluctuations at time $t$ within regions of length-scale much less than $t^{1/z}$ are no longer canonical equilibrium fluctuations. The above scaling form for the two-point functions is likely to hold beyond the model that we considered here.

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